Infection load structured SI model with exponential velocity and external source of contamination

Antoine Perasso, Ulrich Razafison

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Antoine Perasso†, Ulrich Razafison‡

***Abstract*—A mathematical SI model is developed for the dynamics of a contagious disease in a closed population with an external source of contamination. We prove existence and uniqueness of a non-negative mild solution of the problem using semigroup theory. We finally illustrate the model with numerical simulations.**

***Index Terms*—Epidemiology, SI model, nonlinear PDE, transport equation, semigroup theory**

I. Introduction

In this article is considered an infection load epidemiological SI model, described by a system of nonlinear partial differential equations of transport type. The time variable is denoted $t \geq 0$ and the infection load $i \in J = (i^-, +\infty) \subset \mathbb{R}^+$. It is supposed that the infection load i increases exponentially with time according to the evolution equation $\frac{di}{dt} = \nu i$. This leads to the following problem,

$$\begin{cases} \dfrac{dS(t)}{dt} = \gamma - (\mu_0 + \alpha)S(t) - \beta S(t)\mathcal{T}(I)(t), \quad t \geq 0, \\ \dfrac{\partial I(t,i)}{\partial t} = -\dfrac{\partial(\nu i I)(t,i)}{\partial i} - \mu(i)I(t,i) \\ \qquad\qquad +\Phi(i)\beta S(t)\mathcal{T}(I)(t), \quad t \geq 0,\, i \in J, \\ \nu i^- I(t,i^-) = \alpha S(t), \\ S(0) = S_0 \in \mathbb{R}_+, \quad I(0,\cdot) = I_0 \in L^1_+(J). \end{cases} \tag{1}$$

In Problem (1), $\mathcal{T}$ is the integral operator defined for some integrable function h on J by

$$\mathcal{T} : h \mapsto \int_J h(i)\ di,$$

implying that $S(t) + \mathcal{T}(I)(t)$ denotes the total population at time $t \geq 0$, with initial population $S_0 + \mathcal{T}(I_0)$.

Throughout the article the following assumptions are made on the model:

(i) $\beta, \mu_0, \nu, \alpha > 0$ and $\gamma \geq 0$,

(ii) function $\Phi \in C^\infty(J)$ is a non negative function such that $\lim_{i \to +\infty} \Phi(i) = 0$ and $\int_J \Phi(i)di = 1$.

(iii) function $\mu \in L^\infty(J)$ is such that $\mu(i) \geq \mu_0$ for almost every (f.a.e) $i \in J$.

This mathematical model is a variation of a SI epidemiological model of scrapie [8], [10], where the age structured is avoid. See [9] and references therein for a review of SI models described by transport equations, and [3], [4] or [6] for a presentation and examples of classical SI models. Problem (1) describes the dynamics of a contagious disease in a closed population with an external source of contamination. This incorporates infection load structure of the infected population, denoted $i \in J$, with i^- as minimal infection load in the infected population: this infection load i^- is a threshold from which the individual are considered to be infected. As a consequence, an individual with an infection load $i \in (0, i^-)$ appears in the model in the susceptible class S. The model also incorporates a constant mortality rate μ_0 and a constant entering flux γ into the susceptible class S. The mortality rate $\mu(i)$ for the infected class depends on the infection load i. A consequence of the assumption (iii) is that function μ satisfies

$$\lim_{i \to +\infty} \int_{i^-}^{i} \mu(s)ds = +\infty. \tag{2}$$

The limit in equation (2) models that infected individuals leave the stage I by dying of the disease with a finite infection load. The horizontal transmission, with rate β, is modeled with variable initial load of the infectious agent at the contagion, which is assigned using the function Φ. The external contamination is modeled as an input of the system that affects the susceptible with a constant rate α, attributing the minimal initial infection load i^-. This is stated in Problem (1) by the loopback boundary condition $\nu i^- I(t,i^-) = \alpha S(t)$. As a consequence, a zero value of α induces a problem without external contamination.

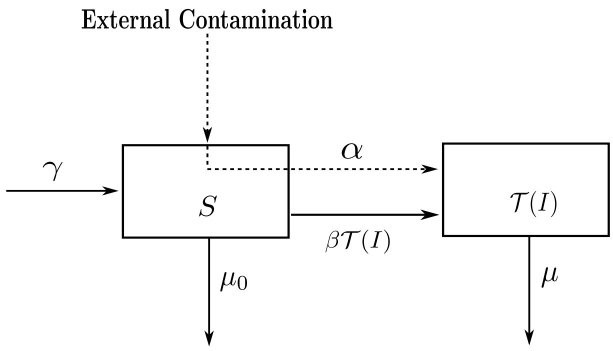

Fig. 1. Fluxes of population dynamics diagram

This article firstly investigates in Section II the well-posedness of Problem (1): the existence and uniqueness of a non-negative mild solution is proved using a semigroup approach. To achieve that goal, we start by checking the existence of a strongly continuous semigroup for the linearized problem in Section II-A, by incorporating the loopback boundary condition in the domain of a densely defined differential operator. Then Section II-B is dedicated to the study of the nonlinear part of Problem (1), proving that this latter satisfies a Lipschitz regularity. This lipschitz perturbation of the linear problem then induces the existence

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and uniqueness of a non-negative mild solution for the nonlinear problem, which is finally proved to be defined on the time horizon $[0,+\infty[$.
In a second step, in Section III, we illustrate the model with numerical simulations throughout a numerical scheme adapted to the model we make explicit in the article.
Finally, in Section IV, we conclude the present work.

II. Mathematical analysis

In all that follows, Δ denotes the set

$$\Delta=\{\lambda\in\mathbb{R},\ \lambda>\nu-\mu_0\},$$

$(X,\|\cdot\|_X)$ is the Banach space with product norm given by

$$X=\mathbb{R}\times L^1(J),$$

and X_+ is the non-negative cone of X, that is $X_+=\mathbb{R}_+\times L^1_+(J)$.
For every constant $R>0$, B_R denotes the ball of X,

$$B_R=\{x\in X,\|x\|_X\leq R\}.$$

A. The linear problem

Related to Problem (1), we consider the differential operator $A:D(A)\subset X\to X$ defined by

$$D(A)=\{(x,\varphi)\in X,\ (i\varphi)\in W^{1,1}(J)\text{ and }\varphi(i^-)=\alpha x\},$$

$$A=\left(\begin{array}{cc}-\mu_0-\alpha & 0\\ 0 & L\end{array}\right),$$

with

$$L\varphi=-\frac{d}{di}(\nu i\varphi)-\mu\varphi.$$

The aim of this section is to prove that $(A,D(A))$ generates a positive C_0 semigroup.

Proposition 1. *The domain $D(A)$ is a dense subset of X, and the resolvent set $\rho(A)$ contains Δ. Moreover, the resolvent R_λ is given for every $\lambda\in\Delta$ by*

$$R_\lambda(y,g)=\left(\begin{array}{c}R_{1,\lambda}(y)\\ R_{2,\lambda}(y,g)\end{array}\right),\tag{3}$$

where

$$\begin{aligned}R_{1,\lambda}(y)&=\frac{1}{\lambda+\mu_0+\alpha}y,\\ R_{2,\lambda}(y,g)&=\frac{\alpha}{\nu i}R_{1,\lambda}(y)e^{-\int_{i^-}^{i}\frac{\lambda+\mu(r)}{\nu r}dr}\\ &\quad+\frac{1}{\nu i}\int_{i^-}^{i}e^{-\int_s^i\frac{\lambda+\mu(r)}{\nu r}dr}g(s)\ ds.\end{aligned}$$

Proof: Consider for every $x\in\mathbb{R}$ the dense subset D_x of $L^1(J)$ given by

$$D_x=\{g\in C_c(\bar{J}),\ g(i^-)=\alpha x\},$$

where $C_c(\bar{J})$ denotes the set of continuous functions with compact support. We clearly have $\bigcup\limits_{x\in\mathbb{R}}\big(\{x\}\times D_x\big)\subset D(A)$, and since $\overline{\cup_{x\in\mathbb{R}}\{x\}\times D_x}=X$, we deduce that $D(A)$ is dense in X.

For $(y,g)\in X$, let us look for $(x,\varphi)\in D(A)$ such that $(\lambda I-A)(x,\varphi)=(y,g)$. This is clearly equivalent to

$$\begin{aligned}x&=\frac{1}{\lambda+\mu_0+\alpha}y,\\ \frac{d\hat{\varphi}}{di}&+\frac{(\lambda+\mu)}{\nu i}\hat{\varphi}=g,\end{aligned}\tag{4}$$

where $\hat{\varphi}(i)=\nu i\varphi(i)$. An integration of the previous equality gives for $\iota\in J$ and $i\geq\iota$,

$$\hat{\varphi}(i)=\hat{\varphi}(\iota)e^{-\int_\iota^i\frac{\lambda+\mu(r)}{\nu r}dr}+\int_\iota^i e^{-\int_s^i\frac{\lambda+\mu(r)}{\nu r}dr}g(s)\ ds.$$

Since we want $(x,\varphi)\in D(A)$, when ι goes to i^- one deduces that φ satisfies

$$\varphi(i)=\frac{\alpha x}{\nu i}e^{-\int_{i^-}^i\frac{\lambda+\mu(r)}{\nu r}dr}+\frac{1}{\nu i}\int_{i^-}^i e^{-\int_s^i\frac{\lambda+\mu(r)}{\nu r}dr}g(s)\ ds.\tag{5}$$

We now prove that such $(x,\varphi)\in D(A)$. Indeed, using the expression of φ given in (5) and assumption (iii) on μ, classical majorations and Fubini's theorem imply for $\lambda\in\Delta$,

$$\begin{aligned}\int_{i^-}^{+\infty}|\varphi(i)|\ di&\leq\frac{\alpha x}{\lambda+\mu_0}\\ &+\int_{i^-}^{+\infty}\left(\int_s^{+\infty}\frac{1}{\nu i}e^{-\int_s^i\frac{\lambda+\mu(r)}{\nu r}dr}\ di\right)|g(s)|\ ds.\end{aligned}\tag{6}$$

For $\lambda\in\Delta$ equation (4) implies $\alpha|x|\leq|y|$ so we deduce from (6)

$$\int_{i^-}^{+\infty}|\varphi(i)|\ di\leq\frac{1}{\lambda+\mu_0}(|y|+\|g\|_{L^1}).\tag{7}$$

This finally implies that $(x,\varphi)\in X$ and consequently to (4),

$$\|(x,\varphi)\|_X\leq\frac{2}{\lambda+\mu_0}\|(y,g)\|_X.$$

We now check that $(i\varphi)\in W^{1,1}(J)$.
Assumption (iii) on μ implies that for $\lambda\in\Delta$,

$$\int_{i^-}^{+\infty}e^{-\int_{i^-}^i\frac{\lambda+\mu(r)}{\nu r}dr}di\leq\int_{i^-}^{+\infty}\left(\frac{i}{i^-}\right)^{-\frac{\lambda+\mu_0}{\nu}}di<+\infty.$$

Moreover, Fubini's theorem and assumption (iii) on μ yield for $\lambda\in\Delta$,

$$\begin{aligned}&\int_{i^-}^{+\infty}\int_{i^-}^{i}e^{-\int_s^i\frac{\lambda+\mu(r)}{\nu r}dr}|g(s)|ds\ di\\ &\leq\int_{i^-}^{+\infty}\left(\int_s^{+\infty}\left(\frac{i}{s}\right)^{-\frac{\lambda+\mu_0}{\nu}}di\right)|g(s)|\ ds\\ &\leq\frac{\nu}{\lambda+\mu_0-\nu}\|g\|_{L^1(J)}.\end{aligned}$$

Equation (5) and the previous estimations prove that $(i\varphi)\in L^1(J)$. Finally, form the expression (5) it is clear that $(i\varphi)\in W^{1,1}(J)$. So $(x,\varphi)\in D(A)$ and the expression (3) of R_λ follows from (4) and (5). ■

Corollary 1. *The resolvent R_λ satisfies*

$$\|R^n_\lambda\|\leq\frac{2}{(\lambda+\mu_0)^n},\quad\forall\lambda\in\Delta,\quad\forall n\in\mathbb{N}^*.\tag{8}$$

Proof: Let us denote $R^n_\lambda=(R^n_{1,\lambda},R^n_{2,\lambda})$ for every $n\in\mathbb{N}$. Using equation (4) and the same calculation we developped to get (7), an induction proves that for every $n \in \mathbb{N}^*$ and every $(y, g) \in X$,

$$|R^n_{1,\lambda}(y)| \leq \frac{1}{(\lambda+\mu_0)^n}|y|,$$

$$\int_{i^-}^{i^+} |R^n_{2,\lambda}(x,y)| \; di \leq \frac{1}{(\lambda+\mu_0)^n}(|y| + \|g\|_{L^1}),$$

and (8) directly yields. ■

Theorem 1. *The differential operator $(A, D(A))$ is an infinitesimal generator of a strongly continuous positive semigroup $\{T_A(t)\}_{t\geq 0}$ on X that satisfies*

$$\|T_A(t)\| \leq 2\, e^{(\nu-\mu_0)t} \quad \forall t \geq 0. \tag{9}$$

Proof: For $\lambda \in \Delta$ one gets $(\lambda+\mu_0-\nu)^n \leq (\lambda+\mu_0)^n$ for every $n \in \mathbb{N}$. Then the Corollary 1 and the Hille-Yosida theorem [5] prove the existence of the semigroup $\{T_A(t)\}_{t\geq 0}$ and the majoration (9). Moreover, as it is proved in proved in [1], the resolvent R_λ being positive on $L^1(J)$, the semigroup $\{T_A(t)\}_{t\geq 0}$ is also positive. ■

B. The non-linear problem

In this section, we tackle the non-linearity in Problem 1 proving it satisfies a Lipschitz condition. To this goal, we check that Problem 1 rewrites as

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} = A \begin{pmatrix} S(t) \\ I(t) \end{pmatrix} + f(S(t), I(t)), \\ S(0) = S_0 \in \mathbb{R}_+, \quad I(0,\cdot) = I_0 \in L^1_+(J), \end{cases} \tag{10}$$

where function $f : X \to X$ is given by

$$f(u,v) = \begin{pmatrix} \gamma - \beta u \mathcal{T}(v) \\ \beta\Phi\, u \mathcal{T}(v) \end{pmatrix}. \tag{11}$$

Lemma 1. *The function $f : X \to X$ given in* (11) *satisfies the following properties :*

1) $\exists \Lambda > 0,\ \forall M > 0,\ \forall ((u_1, v_1), (u_2, v_2)) \in B_M^2$,

$$\|f(u_1,v_1) - f(u_2,v_2)\|_X \leq \Lambda M \|(u_1,v_1) - (u_2,v_2)\|_X,$$

2) $\forall m > 0, \exists \lambda_m > 0,$

$$(u,v) \in B_m \cap X_+ \Rightarrow f(u,v) + \lambda_m(u,v) \in X_+. \tag{12}$$

Proof: Let $M > 0$ and $((u_1, v_1), (u_2, v_2)) \in B_M^2$. Straightforward computations give

$$|u_1\mathcal{T}(v_1) - u_2\mathcal{T}(v_2)| \leq M\|(u_1,v_1) - (u_2,v_2)\|_X.$$

Hypothesis (ii) on Φ and the previous inequality imply

$$\|f(u_1,v_1) - f(u_2,v_2)\|_X \leq \Lambda M \|(u_1,v_2) - (u_2,v_2)\|_X,$$

where $\Lambda = 2\beta$ is a positive constant. Moreover, given $m > 0$, on gets for every $(u,v) \in B_m \cap X_+$ the following estimation,

$$\gamma - \beta u \mathcal{T}(v) + \lambda_m u \geq (\lambda_m - \beta m)u,$$

so (12) is satisfied for every $\lambda_m \geq \beta m$. ■

1) Existence and uniqueness of the solution on finite time horizon: In this section, we aim at proving existence, uniqueness and positivity of the solution of Problem (1) on a finite time horizon. This solution is defined in a mild sense, we refer to [2] for the definition.

Proposition 2. *For every $(S_0, I_0) \in X_+$, there exists $t_{max} \leq +\infty$ such that Problem* (1) *has a unique mild solution $(S, I) \in C([0,T], X_+)$ for every $T < t_{max}$.*

Proof: We prove the theorem with a fixed point method, adapting the ideas of [11].
Let $m > 0$. Consider, for λ_m that satisfies (12), the operator $A_m = A - \lambda_m I$ and the function $f_m = f + \lambda_m I - \begin{pmatrix} \gamma \\ 0 \end{pmatrix}$.
A consequence of Theorem 1 is that A_m is an infinitesimal generator on X of a positive C_0 semigroup $\{T_{A_m}(t)\}_{t\geq 0}$ that satisfies

$$\|T_{A_m}(t)\| \leq 2e^{(\nu-\mu_0-\lambda_m)t}, \quad \forall t \geq 0,$$

so one can consider $m > 0$ big enough such that $r_m > 0$ given by

$$r_m = 2\|(S_0, I_0)\|_X \sup_{t\in[0,1]} \|T_{A_m}(t)\|.$$

satisfies

$$r_m \leq m.$$

In all that follows, let us denote $X_+^{r_m}$ the subset of X given by

$$X_+^{r_m} = X_+ \cap B_{r_m}.$$

Since $r_m \leq m$ we have

$$X_+^{r_m} \subset B_m. \tag{13}$$

Let $\tau > 0$ be such that

$$\tau \leq \min\left(1, \frac{\|(S_0, I_0)\|_X}{r_m(\Lambda r_m + \lambda_m)}\right), \tag{14}$$

where Λ is given in Proposition 1.
Consider the mapping $F : C([0,\tau], X) \to C([0,\tau], X)$ defined by

$$\begin{aligned} F(u(s), v(s)) =& T_{A_m}(t)(S_0, I_0) \\ &+ \int_0^t T_{A_m}(t-s) f_m(u(s), v(s)) \; ds. \end{aligned}$$

Since $f_{\lambda_m}(0) = 0$ in X, Proposition 1 implies that for $t \in [0, \tau]$ and $(u, v) \in C([0,\tau], B_{r_m})$,

$$\begin{aligned} \|F(u(t), v(t))\|_X \leq \sup_{s\in[0,t]} \|T_{A_m}(s)\| (\|S_0, I_0)\|_X \\ + t r_m(\Lambda r_m + \lambda_m)), \end{aligned}$$

and consequently to (14) the mapping F preserves $C([0,\tau], B_{r_m})$. Moreover, equations (12) and (13) imply that F preserves $C([0,\tau], X_+^{r_m})$ for $(S_0, I_0) \in X_+$ since the semigroup $\{T_{A_m}(t)\}_{t\geq 0}$ is positive.
Similar calculations prove that F is a contraction mapping of $C([0,\tau], X)$ with Lipschitz constant $\frac{1}{2}$.
Consequently, F is a contraction of $C([0,\tau], X_+^{r_m})$ and the Banach fixed point theorem implies the existence and the uniqueness of $(\bar{u}, \bar{v}) \in C([0,\tau], X_+^{r_m})$ such that $F(\bar{u}, \bar{v}) = (\bar{u}, \bar{v})$ in $C([0,\tau], X)$. By similar arguments than ones developed in [7], the solution can then be extended on $[0, t_{max}[$

with $t_{max} \le +\infty$.
Finally, every mild solution of Problem (1) is a mild solution of the following problem,

$$\begin{cases} \dfrac{d}{dt}\begin{pmatrix} S(t) \\ I(t) \end{pmatrix} = A_m \begin{pmatrix} S(t) \\ I(t) \end{pmatrix} + f_m(S(t), I(t)), \\ S(0) = S_0 \in \mathbb{R}_+, \quad I(0,\cdot) = I_0 \in L^1_+(J), \end{cases}$$

so the unique fixed point $(\bar{u}, \bar{v})$ of F is also the unique mild solution of Problem (10). ∎

2) Global existence: We now prove that the solution can be extended on the whole horizon time $\mathbb{R}^+$.

Theorem 2. *For every $(S_0, I_0) \in X_+$, the Problem (1) has a unique mild solution $(S, I) \in C(\mathbb{R}^+, X_+)$.*

Proof: We suppose by contradiction that $t_{max} < +\infty$. Then Proposition 2 implies that $(S, I) \in C([0, t_{max}[, X_+)$, and standard results from [7] imply that the solution then satisfies

$$\lim_{t \to t_{max}} \|(S(t), I(t))\|_X = +\infty. \tag{15}$$

Since S and I are non-negative functions and all the parameters are positive, Problem (1) implies that

$$0 \le S(t) \le S_0 + \gamma t, \quad \forall t \ge 0.$$

But since $t_{max} < +\infty$ one can deduce that

$$0 \le \liminf_{t \to t_{max}} S(t) \le \limsup_{t \to t_{max}} S(t) < +\infty. \tag{16}$$

Then equation (15) necessarily implies

$$\limsup_{t \to t_{max}} \|I(t)\|_{L^1(J)} = +\infty. \tag{17}$$

Suppose now that

$$\limsup_{t \to t_{max}} S(t)\mathcal{T}(I)(t) = +\infty. \tag{18}$$

Since from the equation in S of Problem (1) one gets

$$S(t) \le S_0 + \gamma t - \beta \int_0^t S(s)\mathcal{T}(I)(s)\, ds,$$

the latter equality combined to (18) and Fatou's Lemma would imply that $\liminf_{t \to t_{max}} S(t) = -\infty$, which contradicts (16). So the $\limsup$ in (18) is finished. Taking (16)-(17) into account one deduces that necessarily $\lim_{t \to t_{max}} S(t) = 0$ and also $\lim_{t \to t_{max}} S'(t) = 0$. Assigning these limits in the equation in S in Problem (1) one gets

$$\lim_{t \to t_{max}} S(t)\mathcal{T}(I)(t) = \frac{\gamma}{\beta}. \tag{19}$$

Consider now the change of variables $\psi = (\psi_1, \psi_2)$ given by

$$\psi : (t, \xi) \mapsto (t, i) = (t, i^- e^{\nu(t-\xi)}).$$

Classical differential calculus applied to $I \circ \psi$ leads to the following differential equation,

$$\frac{\partial (I \circ \psi)}{\partial t} = -(\mu \circ \psi_2 + \nu)\, I \circ \psi + \Phi \circ \psi_2\, \beta S \mathcal{T}(I).$$

Since $t \mapsto S(t)\mathcal{T}(I)(t)$ is a continuous function, then, taking into account (19) and hypothesis (ii) on function Φ, there exists a positive constant $c > 0$ such that the latter equation implies

$$\left| \frac{\partial (I \circ \psi)}{\partial t} \right| \le c + (\mu \circ \psi_2 + \nu)\, I \circ \psi$$

and so

$$|I \circ \psi(t, \xi)| \le I \circ \psi(0, \xi) + ct + \int_0^t (\mu \circ \psi(s, \xi) + \nu) I \circ \psi(s, \xi) ds.$$

A standard Gronwall inequality argument then gives

$$|I \circ \psi(t, \xi)| \le I \circ \psi(0, \xi) + ct$$
$$+ \int_0^t (I \circ \psi(0, \xi) + cs)(\mu \circ \psi(s, \xi) + \nu) e^{\int_s^t (\mu \circ \psi(u, \xi) + \nu) du} ds$$

But if $t_{max} < +\infty$, then hypothesis (iii) on function μ and the previous inequality yields a contradiction with (17).
To conclude, we necessarily have $t_{max=+\infty}$. ∎

III. Numerical simulations

In this section we illustrate the model with some numerical simulations. We start with the presentation of the scheme.

A. Numerical scheme

We introduce an infection load-time grids where the infection load and the time steps are Δi and Δt respectively. We define $i_{j+1/2} = i^- + j\Delta i$, $t^n = n\Delta t$ and the cells $K_j =]i_{j-1/2}, i_{j+1/2}[$ centered at $i_j = \frac{1}{2}(i_{j-1/2} + i_{j+1/2})$, $1 \le j \le M$ where M is the number of cells . We denote by I_j^n the approximation of the average of $I(t^n, i)$ over the cell K_j, namely

$$I_j^n \simeq \frac{1}{\Delta i} \int_{i_{j-1/2}}^{i_{j+1/2}} I(t^n, i) di.$$

Since the propagation speed of the transport equation is not finite, we use an implicit upwind finite volume scheme in order to compute I_j^n. The general scheme is as follow:

- We compute the initial states:

$$S^0 = S(0) \text{ and } I_j^0 = \frac{1}{\Delta i} \int_{i_{j-1/2}}^{i_{j+1/2}} I_0(i) di.$$

- Assume now S^n and $I^n = (I_1^n, \ldots, I_M^n)$ are computed,

  ▷ we define

$$\mathcal{T}_M(I^n) = \Delta i \sum_{j=1}^{M} I_j^n,$$

  ▷ we compute

$$S^{n+1} = \frac{1}{1 + \Delta t(\mu_0 + \alpha + \beta \mathcal{T}_M(I^n))} (\gamma \Delta t + S^n),$$

  ▷ we compute I^{n+1} by solving the following linear system:

$$- \nu \frac{\Delta t}{\Delta i} i_{j-1/2} I_{j-1}^{n+1} + \left(1 + \nu \frac{\Delta t}{\Delta i} i_{j+1/2} + \Delta t \mu_j\right) I_j^{n+1}$$
$$= I_j^n + \Delta t \Phi_j \beta S^{n+1} \mathcal{T}_M(I^n), \; 1 \le j \le M,$$

  where $\mu_j = \mu(i_j)$ and $\Phi_j = \Phi(i_j)$.

TABLE I
PARAMETER VALUES USED FOR THE SIMULATIONS.

Parameter definition	symbol	value
initial susceptible population size	S_0	100 indiv.
initial infected population size	I_0	0 indiv.
susceptible mortality rate	μ_0	0.1 year^{-1}
infected mortality rate	μ	0.15 year^{-1}
infection load growth rate	ν	10^{-3} year^{-1}
contamination rate	α	0.02 year^{-1}
horizontal transmission rate	β	3.10^{-3} (indiv. year)$^{-1}$
entering flux	$\{\gamma_1;\gamma_2\}$	$\{0;1\}$ indiv.year^{-1}

B. Numerical simulations

For the simulations, we consider the truncated domain (i^-, i^+) where we set $i^- = 1$ and $i^+ = 2$. We use the infection load step $\Delta i = 0.05$ and a time step $\Delta t = 0.1$. We present two cases of simulation. Both suppose that the initial population does not contain infected, stated by $I_0 = 0$.
The first case of simulation corresponds to a zero entering flux in the population ($\gamma_1 = 0$). One can then check on Figure 2 that the total population decreases and converges to 0 with time.
In the second case, the entering flux is not zero ($\gamma_2 = 1$). One can check that, with the parameters used for the simulation, an epidemic occurs at the beginning of the contamination process. Moreover, the disease seems to be persistent persistent in time in the following sense : there exists $\varepsilon > 0$ such that $\liminf_{t\to+\infty} \mathcal{T}(I)(t) \geq \varepsilon$.

IV. CONCLUSION

In this article, we have proved the existence and the uniqueness of a non negative mild solution for a SI model that describes the evolution of a disease in a closed population. This disease is characterized by an exponential velocity of the infection load, a contagious process between individuals, and an external source of contamination. This last is supposed to be proportional to the susceptible population and is modeled with a loopback boundary condition. Accordingly to the simulations made, further investigations on this model shall prove the persistence of the disease when the entering flux γ is non zero.

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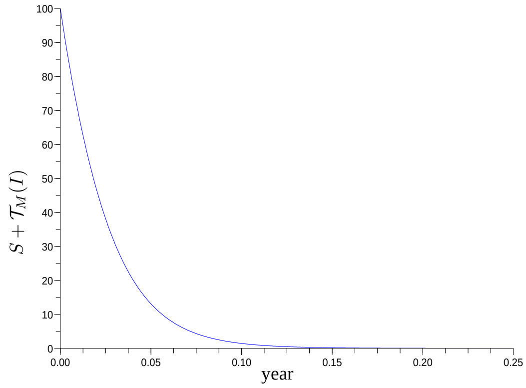

Fig. 2. Case 1 ($\gamma_1 = 0$) ; Total population $S(t) + \mathcal{T}_M(I)(t)$ on a quarter of year

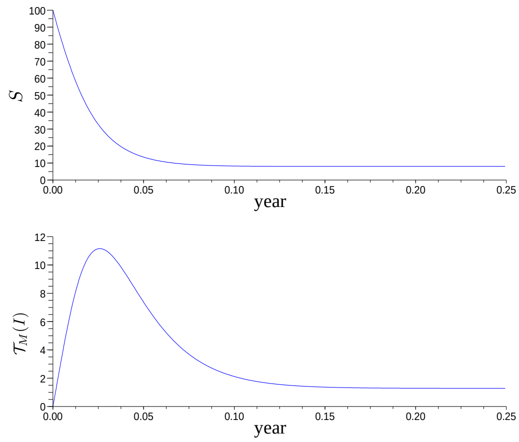

Fig. 3. Case 2 ($\gamma_2 = 1$) ; Susceptible $S(t)$ and total infected $\mathcal{T}_M(I)(t)$ on a quarter of year

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